

All trees contain a large induced subgraph having all degrees $1 \pmod k$

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Abstract

We prove that, for integers $n \geq 2$ and $k \geq 2$, every tree with n vertices contains an induced subgraph of order at least $2[(n + 2k - 3)(2k - 1)]$ with all degrees congruent to 1 modulo k . This extends a result of Radcliffe and Scott, and answers a question of Caro et al.

1. Introduction

An old result of Gallai (see [3, Problem 5.17]) asserts that for every graph G there is a vertex partition $V(G) = V_1 \cup V_2$ such that the induced subgraphs $G[V_1]$ and $G[V_2]$ have all degrees even; it follows immediately that every graph of order n has an induced subgraph with all degrees even with order at least $n/2$. Given a graph G , it is natural to ask for the maximal order $f_2(G)$ of an induced subgraph of G with all degrees odd. It has been conjectured (see [1]) that there is a constant $c > 0$ such that every graph G without isolated vertices satisfies $f_2(G) \geq c|G|$. Let $f_2(n) = \min\{f_2(G) : |G| = n \text{ and } \delta(G) \geq 1\}$. Caro [1] proved $f_2(n) \geq c\sqrt{n}$, for $n \geq 2$, and Scott [6] proved that $f_2(n) \geq n/900 \log n$.

The conjecture has been proved for some special classes of graph (see [1,6]). Caro et al. [2] proved a result for trees and conjectured a better bound. Radcliffe and Scott [4]

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proved the best possible bound,

$$f_2(T) \geq 2 \left\lfloor \frac{|T| + 1}{3} \right\rfloor,$$

for every tree T .

In this paper we consider trees but address the more general problem of determining $f_k(T)$, the maximal order of an induced subgraph of T with all degrees congruent to $1 \pmod k$. This problem was raised by Caro et al. [2], who proved that

$$f_k(T) \geq \frac{2(|T| - 1)}{3k}$$

for every tree T , and conjectured that

$$f_k(T) \geq \frac{|T| + 2k - 4}{k - 1}.$$

This conjecture is not correct however. Here we prove the following best possible bound.

Theorem 1. *For every tree T and every integer $k \geq 2$ there is a set $S \subset V(T)$ such that*

$$|S| \geq 2 \left\lfloor \frac{|T| + 2k - 3}{2k - 1} \right\rfloor$$

and $|\Gamma(x) \cap S| \equiv 1 \pmod k$ for every $x \in S$. This bound is best possible for all values of $|T|$.

We remark that, for $k = 2$, this is the result of Radcliffe and Scott [4] mentioned above; this theorem therefore generalizes that result. Further, results concerning induced subgraphs mod k can be found in [5].

2. Proof of Theorem 1

In this section we give a proof of Theorem 1. The result for $k = 2$ is proved in [4]; we may therefore assume that $k \geq 3$.

We begin by showing that the asserted bound is best possible. Let S_a be a star with $a + 1$ vertices (i.e. the central vertex has degree a), and let $C_{a,b}$ be the graph obtained by taking an S_{a-1} and an S_{b-1} , and joining their centres by an edge (thus $C_{a,b}$ is a rather short caterpillar with $a + b$ vertices). It is immediate to check that, for $a, b \leq k$, the graph $C_{a,b}$ is extremal for the theorem. Larger extremal examples can be obtained by taking $C_{a,b}$ (with $a, b \leq k$) together with any number of copies of $C_{k,k}$ and identifying one endvertex from each graph.

We turn now to the proof that the lower bound holds. Define

$$f(n) = 2 \left\lfloor \frac{n + 2k - 3}{2k - 1} \right\rfloor. \quad (1)$$

For a tree T , we say that $S \subset V(T)$ has *good degrees in T* if the subgraph of T induced by S has all degrees congruent to 1 mod k , and that S is *good in T* if S has good degrees in T and $|S| \geq f(|T|)$.

We use a similar approach to that used in [4]. We suppose that T is a minimal counterexample to the assertion of Theorem 1; it is readily checked that $\text{diam}(T) \geq 4$. Let W_0 be the set of endvertices of T , let W_1 be the set of endvertices of $T \setminus W_0$ and let W_2 be the set of endvertices of $T \setminus (W_0 \cup W_1)$. For $i = 0, 1, 2$ and $v \in V(T)$, let $\Gamma_i(v) = \Gamma(v) \cap W_i$ and let $d_i(v) = |\Gamma_i(v)|$.

We begin with two lemmas giving general useful facts about f_k and f . The lemmas which follow will tighten our grip on the structure of T until it is squeezed out of existence.

Lemma 2. For positive integers n and a_1, \dots, a_n , we have

$$\sum_{i=1}^n f(a_i) \geq f\left(\left(\sum_{i=1}^n a_i\right) - n + 1\right).$$

Proof. Straightforward calculation. \square

Lemma 3. For all $a > k$ we have $f_k(S_a) \geq f(|S_a| + k) + 2$. For $1 \leq a \leq k - 1$ we have $f_k(S_a) = 2 \geq f(|S_a| + k)$. Also $f_k(S_k) = 2 = f(|S_k| + k - 1) = f(2k)$.

Proof. Follows easily from $f_k(S_a) = k \lfloor (a - 1)/k \rfloor + 2$ and (1), since $|S_a| = a + 1$. \square

Lemma 4. Suppose that $x \in W_2$ and set $a = d_0(x)$, $b = d_1(x)$ and $c = |\{v \in \Gamma_1(x) : d_0(v) = k\}|$. Then

$$b(k - 1) \leq a + c.$$

Moreover, if $b(k - 1) = a + c$ then $d_0(v) \leq k$ for all $v \in \Gamma_1(x)$.

Proof. Suppose that $b(k - 1) > a + c$. Write $\Gamma_0(x) = \{v_1, v_2, \dots, v_a\}$ and $\Gamma_1(x) = \{w_1, w_2, \dots, w_b\}$. Renumbering the w_i if necessary, we may suppose that w_1, w_2, \dots, w_c have $d_0(w_i) = k$. Let T_i be the component of $T \setminus x$ containing w_i ($i = 1, 2, \dots, b$) and let T' be the ‘large’ portion remaining. Simply by looking for a good subset S which does not contain x and using Lemmas 2 and 3, we see that

$$\begin{aligned} f_k(T) &\geq f_k(T') + \sum_{i=1}^b f_k(T_i) \\ &= f_k(T') + cf_k(S_k) + \sum_{i=c+1}^b f_k(T_i) \\ &\geq f(|T'|) + cf(|S_k| + k - 1) + \sum_{i=c+1}^b f(|T_i| + k) \end{aligned}$$

$$\begin{aligned} &\geq f\left(|T'| + 2ck + \left(\sum_{i=c+1}^b |T_i|\right) + (bk - c) - b\right) \\ &= f(|T| - a - 1 + b(k - 1) - c), \end{aligned}$$

since $|T| = |T'| + c(k + 1) + \sum_{i=c+1}^b |T_i| + a + 1$. Since, by assumption, $b(k - 1) - a - c - 1 \geq 0$ we have $f_k(T) \geq f(|T|)$, a contradiction. (Recall that T was supposed to be a minimal counterexample to the theorem.)

Furthermore, if we have the equality $b(k - 1) = a + c$, then $d_0(w_i) \leq k - 1$ for $i = c + 1, \dots, b$; otherwise, some T_i has $f_k(T_i) \geq f(|T_i| + k) + 2$ (which again gives $f_k(T) \geq f(|T|)$). \square

Lemma 5. *If $x \in W_2$ then $d_0(x) \leq k - 1$. In fact, if $y \in \Gamma_1(x)$ then*

$$d_0(y) < k \Rightarrow d_0(x) \leq k - 1$$

and

$$d_0(y) \geq k \Rightarrow d_0(x) \leq k - 2.$$

Proof. We begin by proving the first half of the assertion. Suppose on the contrary that $x \in W_2$, $y \in \Gamma_1(x)$, $d_0(x) \geq k$ and $d_0(y) < k$. Let A be any set of k vertices from $\Gamma_0(x)$ and let z be any element of $\Gamma_0(y)$. Set $V_0 = A \cup \Gamma_0(y)$, so $|V_0| \leq 2k - 1$. We can find a good subset S' in $T' = T \setminus V_0$; let

$$S = \begin{cases} S' \cup A, & x \in S', \\ S' \cup \{z, y\}, & x \notin S'. \end{cases}$$

Note that if $x \notin S'$ then also $y \notin S'$. Clearly, S has good degrees in T . Furthermore, we have

$$|S| \geq |S'| + 2 \geq f(|T'|) + 2 = f(|T'| + 2k - 1) \geq f(|T|).$$

Thus, S is good in T , which is a contradiction.

For the second half of the assertion, let us assume that $x \in W_2$, $y \in \Gamma_1(x)$, $d_0(x) > k - 2$ and $d_0(y) \geq k$. We show that this leads to a contradiction.

Let A be any set of $(k - 1)$ vertices from $\Gamma_0(x)$, let B be any set of k vertices from $\Gamma_0(y)$, and let z be any element of B . Let $V_0 = A \cup B$, so $|V_0| = 2k - 1$, and let $T' = T \setminus V_0$. If S' is a good subset of T' then S is a good subset of T , where

$$S = \begin{cases} S' \cup B, & y \in S', \\ S' \cup A \cup B \cup \{y\}, & y \notin S', x \in S', \\ S' \cup \{y, z\}, & x, y \notin S'. \end{cases}$$

This is a contradiction, and we are done. \square

Lemma 6. *If $x \in W_2$ then $d_0(x) = k - 2$ and $d_1(x) = 1$. Furthermore, $d_0(y) = k$, where y is the unique element of $\Gamma_1(x)$.*

Proof. Using the notation of Lemma 4, set $a = d_0(x)$, $b = d_1(x)$ and $c = |\{v \in \Gamma_1(x) : d_0(v) = k\}|$. It follows from Lemma 5 that $a \leq k - 1$, and from Lemma 4 we have $b(k - 1) \leq a + c$. If $a < k - 2$ this inequality has no solutions (since $b > 0$ and $0 \leq c \leq b$). If $a = k - 2$ then we get

$$(b - 1)(k - 1) \leq c - 1, \quad (2)$$

while if $a = k - 1$ then

$$(b - 1)(k - 1) \leq c. \quad (3)$$

The only solution of (2) is $b = c = 1$, which is what was claimed. In (3), however, for $a = k - 1$, there are more possibilities. Let us first consider the general case when $b = 1$, and so $\Gamma_1(x) = \{y\}$, say. Suppose that $d_0(y) \neq k$, and so $c = 0$. Thus, we have equality in (3), and it follows from Lemma 4 that $d_0(y) \leq k$. Therefore, we have $b = 1$, $d_0(x) = k - 1$ and $d_0(y) \leq k - 1$. Set $V_0 = \Gamma_0(x) \cup \Gamma_0(y) \cup \{y\}$. Now $|V_0| \leq 2k - 1$ and by the minimality of T we can find a good subset S' in the tree $T' = T \setminus V_0$. Let z be any element of $\Gamma_0(y)$ and set

$$S = \begin{cases} S' \cup y \cup \Gamma_0(x), & x \in S', \\ S' \cup \{y, z\}, & \text{otherwise.} \end{cases}$$

S is good in T , which is a contradiction. Therefore, we must have $d_0(y) = k$, as asserted.

If $b \neq 1$, the only possibility is the special case $a = b = c = 2$ and $k = 3$. Let y_1 and y_2 be the two elements of $\Gamma_1(x)$, pick z_1 and z_2 with $z_i \in \Gamma_0(y_i)$ and pick $w \in \Gamma_0(x)$. Set $V_0 = \Gamma_0(x) \cup \{y_1, y_2\} \cup \Gamma_0(y_1) \cup \Gamma_0(y_2)$, so $|V_0| = 10$. There is some set $S' \subset T \setminus V_0$ which is good in $T \setminus V_0$. Set

$$S = \begin{cases} S' \cup (V_0 \setminus \{w\}), & x \in S', \\ S' \cup \{y_1, z_1, y_2, z_2\}, & x \notin S'. \end{cases}$$

Then S has good degrees in T and $|S| \geq |S'| + 4 \geq f(|T| - 10) + 4 = f(|T|)$. Thus, S is good in T , which is a contradiction.

So far we have proved that if $a = k - 1$ then $b = 1$ and $d_0(y) = k$; but this contradicts Lemma 5. The only remaining possibility is that asserted in the lemma. \square

We are now ready to finish the proof of Theorem 1. Lemma 6 has given us a great deal of information about the neighbourhood of any $x \in W_2$. Now let $x_0 x_1 x_2 \cdots x_m$ be a path in T of maximal length. Since $\text{diam}(T) \geq 4$ we know that $m \geq 4$ and $x_i \in W_i$ for $i = 0, 1, 2$.

We split the proof into cases according to whether $d_0(x_3) = 0$ or not. If $d_0(x_3) = 0$ then we shall find a large good subset in each component of $T \setminus x_3$. These components consist of: some number, possibly zero, of stars (coming from elements of $\Gamma_1(x_3)$); at least one copy of $C_{k-1, k+1}$, one for each element of $\Gamma_2(x_3)$; and the rest of the tree, say T' . Let S' be a good subset of T' . From each star T'' we can pick a good subset

of size at least $2|T''|/(2k-1)$ and from each caterpillar we can pick a good subset of size $k+2$. Because of the form of f we simply need to ensure that the good subsets which we find have total size at least $2|T \setminus T'|/(2k-1)$. This is clearly achieved in the stars, and more than achieved in the caterpillars, with enough spare to account to x_3 . Thus, the union of S' with these smaller good subsets is a good subset of T .

If $d_0(x_3) > 0$ then we use a slightly different approach. Let v be an element of $\Gamma_0(x_3)$ and consider $V_0 = \Gamma_0(x_1) \cup \Gamma_0(x_2) \cup \{v\}$. Note that $|V_0| = 2k-1$. Let S' be a good subset of $T' = T \setminus V_0$ and let

$$S = \begin{cases} S' \cup \Gamma_0(x_1), & x_1, x_2 \in S', \\ S' \triangle \{v, x_0, x_1, x_2\}, & x_2 \in S', x_1 \notin S', \\ S' \cup \{x_0, x_1\}, & x_1, x_2 \notin S'. \end{cases}$$

Then S has good degrees in T and

$$|S| \geq |S'| + 2 \geq f(|T'|) + 2|V_0|/(2k-1) = f(|T|).$$

Therefore, S is good in T , which contradicts the claim that T is a counterexample to the theorem. We have therefore proved Theorem 1. \square

The problem of determining for a tree T the largest $S \subset V(T)$ such that $T[S]$ has all degrees congruent to 0 modulo k is equivalent to the problem of determining the largest independent set. It would, however, be interesting to give bounds on the size of the largest $S \subset V(T)$ such that all vertices in $S[T]$ have either degree 1 or degree congruent to 0 modulo k .

In general, for graphs with minimal degree sufficiently large, it would also make sense to ask for bounds on the size of the largest induced subgraph with all degrees congruent to i modulo k , where $0 \leq i \leq k$.

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